S.-T. Yau College Student Mathematics Contest, 2018
Applied Mathematics, Individual

1. Consider the definite integral

$$I = \int_a^b f(x)\, dx.$$

   a. Construct an approximation to I by using the composite trapezoidal rule with a uniform partition of the interval $[a, b]$.

   b. Suppose $f \in C^2[a, b]$, show that the above approximation is second-order accurate.

   c. Suppose f is periodic and smooth on the interval $[a, b]$, show that the above approximation is spectral order accurate.

2. Let $(\mathbf{u}, p)$ be the solution of the Stokes equation on a domain Ω

$$\begin{cases} -\nabla p + \Delta \mathbf{u} = 0 \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u}|_{\partial\Omega} = g \end{cases} \tag{1}$$

where $\mathbf{u}(x, y) = (u(x, y), v(x, y))$, $\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and g is the given boundary value. Show that $\mathbf{u}$ is a minimizer of the dissipation functional

$$E(\mathbf{v}) = \int_\Omega |\nabla \mathbf{v}|^2 dxdy$$

among all $\mathbf{v}$ such that $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{v}|_{\partial\Omega} = g$.